

Math Circle - Tetris Pieces and Colorful Chess Wars

January 18, 2024

Look at the color of the tiles!

Dominoes on a broken chessboard

Given a full chessboard (that is, a board of size 8×8), we can fill it up completely with 1×2 dominoes, in such a way that two dominoes never overlap. In the picture below, we can see an example with light blue dominoes.

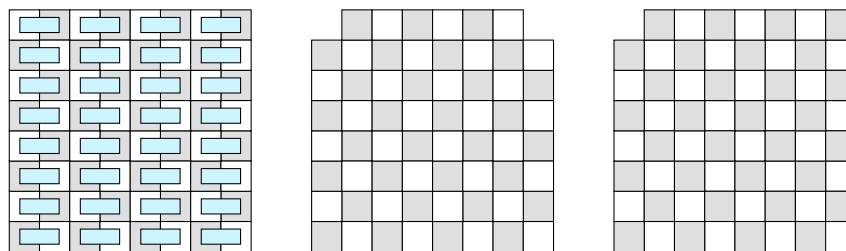


Figure 1: A chessboard covered in dominoes on the left, as well as two chessboards with different pairs of removed corners, respectively adjacent and opposite corners.

Unfortunately, two of our chessboards are broken, in the same way as in the picture. We would like to find out in each situation if it is still possible to fill the board up. If it is possible, we want to find a way to arrange the dominoes. If it isn't, we want to *prove* that it isn't. That is our first example problem:

Example Problem 1

Given a chessboard with two corners removed, is it possible to completely fill it up with 1×2 dominoes, knowing that

- a) The two removed corners are consecutive?
- b) The two removed corners are opposite?

Example Solution 1

After trying to fit in dominoes in each of the boards for a while, one likely reaches the theory that *a)* is possible, whereas *b)* is impossible. While finding a configuration for *a)* is a matter of playing around with dominoes until we get lucky, in item *b)* we need an original idea in order to convince ourselves (and others) that filling up the board truly is impossible. As hinted in the title, this idea is to look at the color of the tiles:

- a) It is possible to fill up the board. For example, we can adapt the picture above by shifting the dominoes in the first row one tile to the right, and removing the fourth one.
- b) It is not possible to fill up the board. Indeed, whenever we place a domino, it will occupy exactly one white tile and one black tile, so the number of white and black tiles occupied by dominoes must be exactly the same. By removing two opposite corners like in the picture above, we were left with 32 gray tiles, but only 30 white tiles, which means that they can't all be filled by dominoes, as 31 dominoes would occupy 31 white tiles and 31 gray tiles.

The Great Knight War

In the country of chess knights, tensions are so high that whenever two knights are in position to attack each other, they will. Recall that in chess, the knights attack in L :

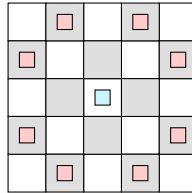


Figure 2: A knight in the blue square attacks exactly the cells with red squares, by chess rules.

The knight leader came to us for help, as he wants to utilize chessboard space as well as possible, while making sure he isn't causing any commotion.

Example Problem 2

What is the maximum number of knights that can fit in a chessboard such that no two knights attack each other?

When we have a problem which asks for a maximum number which satisfies something, we actually have two problems in disguise:

1. Showing that our number works;
2. Showing that no higher number works;

Sometimes, we play around with the problem enough to get a suspicion of the maximum number - typically the best we found while exploring (this gives us part 1 for free) - and then proceed to try to prove part 2. Some other times, we find an argument which works for part 2, and then we try to find a configuration with the number we think is maximum, to get part 1. Regardless of the order, we need to make sure we do both.

Example Solution 2

In this case, we start with an idea, the same idea as the previous problem: looking at the color of the tiles. We can see in figure 2 that a knight only attacks tiles of the opposite color. That means that if we place knights in every white tile, they will not attack each other! That gives us part 1:

1. We can place 32 knights on a chessboard in a way that they don't attack each other, by placing them all in white tiles, which never attack each other.

So, we think the maximum number is 32, which is half of the tiles. In order to show we can't have more knights than that, it is enough to partition the tiles into pairs which attack each other - that way there can be at most one knight in each pair - so there can be at most 32 knights. This gives us part 2 :

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Figure 3: Each number links two cells which attack each other.

2. In figure 3, we can see that the board can be partitioned into 32 pairs of tiles which attack each other. As a result, there can be at most one knight for each pair, which means we can't place more than 32 knights such that no two attack each other.

Questions about the example problems

Here are some questions related to the example problems if you want to further think about them. Bonus points if you come up with your own follow up questions - there are still plenty of them left!

- 1.1. If instead of a chessboard we had a 9×9 board and instead of 1×2 dominoes we had 1×3 triominoes, could we completely fill the board with the triominoes? Is there any trio of corners that can be removed without making it impossible?
- 1.2. Let's say that there is a broken chessboard, but now the fallen tiles are not necessarily the corners. Can you think of a property on the tiles which is *necessary* to still be able to fill it up? Is this property *sufficient* or can you find an example where the property is satisfied yet you still can't place dominoes in every remaining tile without overlaps?
- 2.1. If we had a 4×4 board, would the solution to the knight problem change? Can you find a formula you think works for a $n \times n$ board, with n multiple of 4? What happens to our solution if n is even but not a multiple of 4 - which parts work unchanged and which parts need to be adapted?
- 2.2. Can you find an even value of n - not a multiple of 4 - for which the formula you found on 2.1 does not work? Hint: think of small n , you might find what one calls a *degenerate* case.
- 2.3. [Challenge; to try at home] Can you solve the problem for any $n \times n$ board?

Add colors to the tiles!

In the two examples above, the colors that come with chessboards proved very useful in solving problems involving them. In a lot of cases, we get blank boards, and it is our job to find colorings that help us solve the problem. Here are some helpful colorings for inspiration:

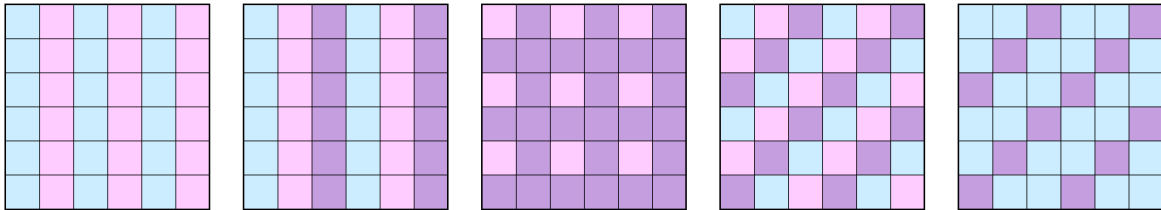


Figure 4: Some examples of colorings (definitely not an extensive list!).

Extra Problems

In the problems that follow, covering a board using certain types of pieces is always assumed to not allow overlapping pieces or pieces (even if partially) leaving the board.

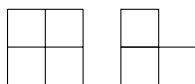
1. We want to cover a 6×6 board with single a type of tetris piece. For each of the following types, what is the maximum amount of pieces we can place on the board?



2. [Italy Team Selection Test] Given an 8×8 board, we want to remove a tile and be able to completely cover the remaining tiles with 1×3 pieces (they can be rotated). What are the tiles we can remove?



3. We want to paint purple some tiles in a 9×9 board, in such a way that there are not four purple tiles in a row (horizontally nor vertically). What is the maximum amount of tiles we can paint purple?
4. [Portugal Team Selection Test] Find the smallest value of n for which we can cover a $n \times n$ board using the same amount of each of the following pieces:



5. Can we completely cover a 10×10 board using the piece below?

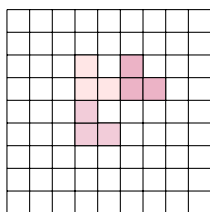


6. [Olimpiada de Mayo]

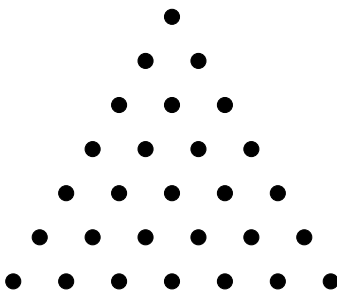
- a) Can we completely cover a 9×9 board with the piece below?



- b) If the following pieces are already placed, can we finish covering the board?



7. [Olimpiada de Mayo] We have 28 points in a lattice as shown below. An operation consists in choosing 3 points which are the vertex of a common equilateral triangle and removing them. These operations are performed until there is only one point left in the lattice. What are the possibilities for this final point?



8. [International Mathematical Olympiad, 2018-4¹] Queenie and Horst play a game on a 20×20 chessboard. In the beginning the board is empty. In every turn, Horst places a black knight on an empty square in such a way that his new knight does not attack any previous knights. Then Queenie places a white queen on an empty square. The game gets finished when somebody cannot move. Find the maximal positive K such that, regardless of the strategy of Queenie, Horst can put at least K knights on the board.

¹The problem statement here was taken from the Shortlist version, since the final version avoids mentioning chess. Note that it does not matter if Queenie's queens attack each other or the knights.